Def. Let X be a topological space. Call a collection $\mathcal{B}$ of open subsets of X a

o **prebasis** (also **subbasis**, but I don't like this since "sub" suggests it being a subset of a basis) if $\mathcal{B}$ generates the topology of X.

o **basis** if every nonempty open set in X is an (arbitrary) union of sets from $\mathcal{B}$.

Examples. (a) In a metric space X, open balls form a basis for the topology induced by the metric since by definition a set is open in X if it is a union of open balls.

(b) In a metric space (X, d), the collection
$$\mathcal{B} := \{ B_{\frac{1}{n}}(x) : x \in X \text{ and } n \in \mathbb{N}^+ \}$$
is a basis for X.

Proof. Fix any nonempty open $U \subseteq X$. For each $x \in U$ there is an open ball $B_r(x) \subseteq U$ for some $r > 0$. Let $n_x \in \mathbb{N}^+$ be large enough so that $\frac{1}{n_x} < r$ hence $\underbrace{B_{\frac{1}{n_x}}(x)}_{U_x} \subseteq U$. Then $U = \bigcup_{x \in U} B_{\frac{1}{n_x}}(x)$. $\qquad \square$

(c) In $\mathbb{R}$, bdd open intervals form a basis by example (a) because they are exactly the open balls. The collection
$$\mathcal{H} := \{ (a, \infty), (-\infty, a) : a \in \mathbb{R} \}$$
is not a basis, well, because $(0,1)$ is not a union of sets in $\mathcal{H}$, in fact, no set in $\mathcal{H}$ is contained in $(0,1)$. However, $\mathcal{H}$ is a prebasis since each open interval (a, b) is an intersection of two sets in $\mathcal{H}$, namely, $(a, b) = (-\infty, b) \cap (a, \infty)$.

(d) In $\Sigma^{\mathbb{N}}$ with its usual metric, $\Sigma \neq \emptyset$, the cylinders form a basis because they are exactly the balls in $\Sigma^{\mathbb{N}}$. We now define a prebasis which is not a basis. For each $i \in \mathbb{N}$ and $\sigma \in \Sigma$, put $[i \mapsto \sigma] := \{ x \in \Sigma^{\mathbb{N}} : x(i) = \sigma \}$.

<u>Claim.</u> The collection $\mathcal{E} := \{ [i \mapsto \sigma] : i \in \mathbb{N} \text{ and } \sigma \in \Sigma \}$ is a prebasis.

<u>Proof.</u> Since cylinders form a basis, it is enough to show that $\mathcal{E}$ generates all cylinders. In fact, every cylinder $[w]$, where $w \in \Sigma^{<\mathbb{N}}$, is a finite intersection of sets in $\mathcal{E}$: $[w] = \bigcap_{i \in \ln(w)} [i \mapsto w(i)]$. □

Recall that cylinders are clopen, so we have a basis of clopen sets. A topological space admitting a basis of clopen sets is called <u>0-dimensional</u>. Thus, $\Sigma^{\mathbb{N}}$ is 0-dimensional.

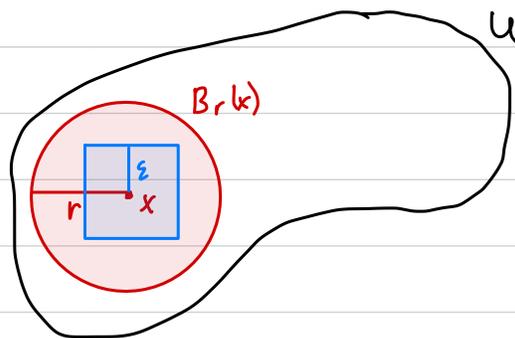(e) In $\mathbb{R}^d$, the collection $\mathcal{B}$ of all open boxes is basis. Recall that an open box is a set of the form $I_1 \times I_2 \times \dots \times I_d$, where each $I_k \subseteq \mathbb{R}$ is an open interval.

<u>Proof.</u> Fix a non-empty open $U \subseteq \mathbb{R}^d$. Then each $x \in U$ has an open ball $B_r(x) \subseteq U$ for some $r > 0$. We choose $\varepsilon > 0$ small enough so that the box $R := \prod_{i=1}^{d} (x_i - \varepsilon, x_i + \varepsilon) \subseteq B_r(x)$. Such an $\varepsilon > 0$ exists because for each $y \in R$,
$$d_2(y, x) = \sqrt{\sum_{i=1}^{d} |y_i - x_i|^2} < \sqrt{\sum_{i=1}^{d} \varepsilon^2} = \sqrt{d} \cdot \varepsilon,$$
so it's enough to take $\varepsilon < r/\sqrt{d}$. □

<u>Convention.</u> An <u>empty intersection</u> of subsets of a set X is defined to be X, i.e. if I is an index set and $X_i \subseteq X$ for each i, then if $I = \emptyset$, then
$$\bigcap_{i \in I} X_i := X.$$

<u>Cor.</u> Let $\mathcal{E}$ be a prebasis for a topological space X.

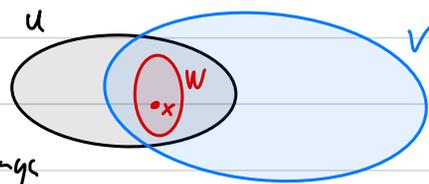(a) The collection $\mathcal{E}_\cap$ of all finite intersections of sets in $\mathcal{E}$ is a basis. Note that $\mathcal{E}_\cap \ni X$ since X is the empty intersection of sets in $\mathcal{E}$.

(b) If $\mathcal{E}$ is closed under finite intersections (and $X \in \mathcal{E}$) then $\mathcal{E}$ is a basis.

<u>Proof.</u> We already proved (a) last time, and (b) follows from (a). □

<u>Characterization of basis.</u> Let X be a top. space. A prebasis $\mathcal{E}$ for X is a basis $\iff$
  for every pair $U, V \in \mathcal{E}$, every $x \in U \cap V$ admits $W \in \mathcal{E}$ such that $x \in W \subseteq U \cap V$.
<u>Proof.</u> $\implies$ If $\mathcal{E}$ is a basis, then every nonempty open set is
  a union of sets in $\mathcal{E}$. In particular, for $U, V \in \mathcal{E}$ s.t. $U \cap V \neq \emptyset$,
  $U \cap V$ is a union of sets in $\mathcal{E}$, so each $x \in U \cap V$ belongs
  to one of the sets W in that union.

$\impliedby$. Suppose the right side. To show that $\mathcal{E}$ is basis, it suffices to show that any finite
  intersection $U := \bigcap_{i < n} U_i$ of sets $U_i \in \mathcal{E}$ is union of sets in $\mathcal{E}$ because the collection
  $\mathcal{E}_{\cap}$ is basis. For this it's enough to show that every $x \in U$ has a set $W_x \in \mathcal{E}$
  with $x \in W_x \subseteq U$ because this would make $\bigcap_{i<n} U_i = \bigcup_{x \in U} W_x$. Fix $x \in U$. We prove by
  induction on n. Suppose the statement is true for $n-1$ and we prove for n.
  Let $U' := \bigcap_{i < n-1} U_i$. Then $x \in U'$ and we already know by induction that $\exists V \in \mathcal{E}$
  such that $x \in V \subseteq U'$. Since $x \in U$, $x \in U_{n-1}$, so $x \in V \cap U_{n-1}$ and both
  $V, U_{n-1} \in \mathcal{E}$. Thus, by the assumption $\exists W_x \in \mathcal{E}$ with $x \in W_x \subseteq V \cap U_{n-1} \subseteq U' \cap U_{n-1} = \bigcap_{i<n} U_i$. $\square$

We now consider an interesting example of a topology on $\mathbb{Z}$ which is Hausdorff and
0-dimensional, using which we will give a funny proof that there are ∞-many primes.
The definition will only use the group structure of $(\mathbb{Z}, +)$ and can be defined more
generally for arbitrary groups.

<u>The profinite topology on $\mathbb{Z}$.</u> Let $\mathcal{E}$ be the collection of all arithmetic progressions in $\mathbb{Z}$,
i.e. sets of the form
$$a + b\mathbb{Z} := \{a + bz : z \in \mathbb{Z}\}$$
where $a, b \in \mathbb{Z}$. Note that $b\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ and $a + b\mathbb{Z}$ is a coset of
that subgroup. All subgroups of $\mathbb{Z}$ are of the form $b\mathbb{Z}$ for some $b \in \mathbb{Z}$, so $\mathcal{E}$ is
simply the collection of all cosets of all subgroups of $\mathbb{Z}$.
Let $\mathcal{T}_p$ be the topology generated by $\mathcal{E}$. This topology is called the <u>profinite</u>
<u>topology on $\mathbb{Z}$</u>, and can be defined for any group as the topology generated by

all cosets of all subgroups, but we will only discuss it for $\mathbb{Z}$ here.

Claim. Σ is closed under finite intersections, and hence is a basis for $\mathcal{T}_p$.
Proof. HW.

Claim. Each set $a + b\mathbb{Z} \in \Sigma$ is clopen, so $\mathcal{T}_p$ is 0-dimensional.
Proof. Indeed, $(a + b\mathbb{Z})^c = \bigcup_{\substack{0 \le c < |b| \\ c \ne a}} (c + b\mathbb{Z})$, hence $(a + b\mathbb{Z})^c$ is open so $a + b\mathbb{Z}$ is closed, hence clopen. $\square$

Claim. $\mathcal{T}_p$ is invariant under translation by any $a \in \mathbb{Z}$ as well as inversion, i.e. if $U \in \mathcal{T}_p$ then $a + U$ and $-U$ are also in $\mathcal{T}_p$.
Proof. Since Σ is a basis, it is enough to check for sets in Σ, so fix $c + d\mathbb{Z} \in \Sigma$. Then $a + (c + d\mathbb{Z}) = (a + c) + d\mathbb{Z} \in \Sigma$ and $b \cdot (c + d\mathbb{Z}) = (b \cdot c) + (b d) \cdot \mathbb{Z} \in \Sigma$. $\square$

Claim. $\mathcal{T}_p$ is Hausdorff.
Proof. For any distinct $a, b \in \mathbb{Z}$, may assume $a < b$ WLOG. By translation invariance, we may assume WLOG that $a = 0$ (replace a and b with $a - a$ and $b - a$). Then $(b+1)\mathbb{Z} \ni a = 0$ and $b + (b+1)\mathbb{Z} \ni b$ and $(b+1)\mathbb{Z} \cap (b + (b+1)\mathbb{Z}) = \emptyset$ because they are distinct cosets of the same subgroup (check yourself). $\square$